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THE LAWS OF DYNAMICS, AND THEIR TREATMENT IN TEXT-BOOKS.

Continued from page 389.

In the preceding discussion of the Galileo-Newton theory, force has been introduced as measured by mass-acceleration, in accordance with the generally accepted procedure. The existence of laws, connecting force with the conditions accompanying it, has been referred to as a fundamental fact which makes the theory possible; though, as a matter of logical order, all details as to such laws have been put into a secondary position. The recognition of the fact that acceleration was the thing, which could be connected by laws with the conditions under which a body moved, has been mentioned as Galileo's great achievement. This was historically the first and most important, and was probably the most difficult, step to be taken. There is no record of any suggestion of it before Galileo's time, and no theory of motion of any value had been constructed on any other basis. Perhaps the nearest approach to such a theory was the study of the celestial motions on the basis of circular motion; and this was not capable of progressing beyond the empirical stage.

There are two general laws of force; the first of which may be briefly referred to as being that under constant conditions force is constant, and the second that, if two sets of conditions are superposed, the forces accompanying them are added. Galileo's chief experimental result was the establishment of the first law approximately, in the case of the approximately constant conditions attending the motion of a falling body; and one great merit of this investigation consisted in the insight which led to the elimination of such disturbance as the resistance of the air. The existence of a disturbance of this character

introduces the natural objection that we find forces depending on the velocities of the moving bodies. But the view is that, in such a case as the resistance of the air, the velocity effect is due to the consequent attitude of the surrounding air, and that the statement in terms of the velocity of the moving body is merely a convenient expression of the way in which such surrounding conditions vary. The second law is commonly stated as the principle of the physical independence of forces. There are certain cases in which the approximate superposition of conditions without interference can be readily arranged, and for such cases the law can be tested. The case in which the conditions imposed upon a small body consist of its attachment to another small body by a uniformly stretched elastic thread may be looked upon as a model of the sort of class here referred to, being one in which the physical surroundings, which appear to be material to the question, can be sufficiently perceived and isolated. It is not easy to specify the scope of the law completely, for it should be noted that, in the case of some special laws of force, such as the law of gravitation, this general law may be considered to be included as part of the special law. The comparison of forces by balancing has already been discussed, as being, where it is applicable, a method of comparison independent of an investigation of what has been called a Newtonian base; and this, combined with the law of addition of forces by superposition of conditions, gives what is commonly known as the statical measure of force. It has been the fashion with some writers to discredit the use of this measure of force; but nevertheless it is not, and is not likely to be, wholly discarded in elementary books, since the provisional use of it permits some progress to be made in the subject without the complications attending the introduction of mass and base.

Before pursuing this point further, let us consider what position we are in when we pass from the Galileo-Newton theory of the motion of particles to the study of the statics of rigid bodies. The step which has to be taken is the establishment of the principle of transmissibility of force, or of something equivalent to it. This cannot be regarded as altogether easy, and by some writers seems to have been rather slurred over. Professor Love, in his *Theoretical Mechanics*, has recognized what is needed for its logical treatment; but his argument, though brief in form, is of a kind which may perhaps not be very easily appreciated by a beginner. The following, though longer, may be regarded as, in some respects, a more direct investigation of the point in question.

The motion of a rigid system of points, relative to a given base, may be specified, at any moment, by the velocity of one point and the angular velocity; that is to say, by six quantities.

Let us exclude the case of the points all lying in one straight line as inapplicable to any actual body. Take any set of rectangular coordinate axes, moving in any manner relative to the base. Let u_0, v_0, w_0 be the components, in the directions of the axes, of the velocity of the point, either belonging to the system, or rigidly attached to it, which happens to be passing through the origin; let p, q, r be the components of the angular velocity of the system; and let u, v, w be the components of the velocity of a point x, y, z of the system. Then

$$u = u_0 + zq - yr$$
$$v = v_0 + xr - zp$$
$$w = w_0 + yp - xq.$$

There are three times as many quantities u, v, w as there are points in the system; and if any six independent linear functions of these quantities are known to be zero, it will follow that u_0, v_0, w_0, p, q, r must all be zero, that is to say all the quantities u, v, w are zero. Let numerical quantities $m_1, m_2 \ldots$ be assigned arbitrarily to the several points of the system; then, if Σmu, Σmv, Σmw, $\Sigma m(yw - zv)$, $\Sigma m(zu - xw)$, $\Sigma m(xv - yu)$ are all zero, it will follow that the velocities are all zero, provided that the six equations thus given for u_0, v_0, w_0, p, q, r are independent. We shall find that this is certainly the case if the m's are all positive.

Let $\Sigma m = M$, $\Sigma mx = M\bar{x}$, etc., then our six equations are of the two types

$$M(u_0 + \bar{z}q - \bar{y}r) = 0,$$
$$M(\bar{y}w_0 - \bar{z}v_0) + p\Sigma m(y^2 + z^2) - q\Sigma mxy - r\Sigma mzx = 0.$$

Assuming that M is not zero, and eliminating u_0, v_0, w_0, we get three equations of the type

$$p\{\Sigma m(y^2 + z^2) - M(\bar{y}^2 + \bar{z}^2)\}$$
$$- q(\Sigma mxy - M\bar{x}\bar{y}) - r(\Sigma mzx - M\bar{z}\bar{x}) = 0.$$

Write these equations in the form

$$Ap - Fq - Er = 0$$
$$-Fp + Bq - Dr = 0$$
$$-Ep - Dq + Cr = 0;$$

eliminating p, q, r we get

$$ABC - 2DEF - AD^2 - BE^2 - CF^2 = 0;$$

but this, in the language of dynamics, is equivalent to saying that the product of the three principal moments of inertia of the system at the centre of gravity is zero; and, if the m's are all positive, this cannot be the case, nor can M be zero.

We can now prove the proposition that a system of external forces (components X, Y, Z), satisfying the six equations of the types $\Sigma X = 0$, $\Sigma(yZ - zY) = 0$, do not affect the motion of a rigid body. To prove this, notice that the internal forces, called into play, will by themselves satisfy equations of the same type, since they occur in pairs; hence the whole system of forces applied to particles of the body satisfies these equations. Such a system of forces implies the generation, in a time dt, of a velocity distribution udt, vdt, wdt relative to a Newtonian base, satisfying the six equations of the types $\Sigma mu = 0$, $\Sigma m(yw - zv) = 0$, where m is positive; that is to say the combined effect of the forces upon the motion is nil. Thus the equations used in statics for a rigid body are proved. It is clear that they are not only sufficient, but also necessary.

The point to be insisted upon is that, if we are to derive the mechanics of rigid bodies logically from a theory of the motion of particles, the theoretical step, of introducing the relations between the motions of the points of a rigid system, is of such a character as to be rather awkward to deal with in an elementary fashion. Thus we are led to make an independent appeal to experiment; and this is an additional reason for beginning elementary statics on a somewhat independent footing. If, by the methods of balancing and superposition, we get a measure of force, by means of which the subject can be developed within a certain range, within which the ideas of mass and Newtonian base have no place, the simplest and most scientific course is to carry out the development within that range without bringing in this machinery. It seems to be possible to maintain, in fact, that the modern procedure of basing the whole of Mechanics, from the beginning, on the laws of motion is inadvisable from the points of view both of simplicity and of logic.

In the definition of any physical quantity we are concerned only with the identification of it and the measurement of it. A definition may embrace several alternative measurements; and these may either be mathematical functions of each other, or may be dependent for consistency upon some physical law. Allowing, if we do so, the definition of force to embrace a superposition measurement, is a case of the latter sort. It appears also that a superposition measurement holds a subordinate and provisional position, for the scope of it is practically limited, so that the subject could not be developed fully with it alone.

Elementary statics might be introduced somewhat as follows. The causal relations between the motions of portions of matter, analysed into relations between the motions of particles, are to be dealt with fully by the Galileo-Newton theory. But, as a preliminary step, the existence of some of these relations may be recognisable without the aid of this theory. The contact at a

point between two solid bodies is obviously a case in which a relation exists, and given cases of such contact can be reproduced with some precision. Without going into the way in which the motions are to be measured, we can study the condition under which they are neutralised. The establishment by statical experiments of the fact that the pressure effect at a point has a direction is not found to be a difficulty, and superposition gives a measurement of magnitude. A quantity with this direction, and with a magnitude thus measured, in terms of any recognisable unit, may be called a force. Other forces may then be recognised by the method of balancing against a certain identifiable pressure.

Let us consider the limitations of the subject as developed in this way. We at once meet with weight, and are led to regard it as a force, but have no means at our disposal for investigating the question of its reciprocal character. So this question must be left in abeyance; and in fact we know that, when studied by the light of a more complete theory of motion, weight is found not to possess this character completely. Moreover, such examples as the relative equilibrium of the governor of a steam engine would introduce centrifugal force in a prominent manner. This could not be reconciled with the idea of mutual relation with which we started, if a more complete theory did not come to the rescue. It seems to be the most natural procedure, in an elementary development of the subject, that a theory of motion with mass and base should come in at this particular stage, the superposition measure of force at the same time giving place to a measure based on acceleration.

The parallelogram of forces makes its appearance in connection with elementary statical experiments; and to make this, at this stage of the subject, dependent on a theory of mass, as appears to be usual in modern text-books, somewhat grates upon one's sense of logical order. It seems rather confusing to hang it on to a crude and provisional theory of the universe, seeing that it is capable of being dealt with without so elaborate a preface. The old-fashioned statical proofs of this proposition seem to give a better view of its true position. Duchayla's proof, which held its own in text-books for many years, has something to be said for it, in spite of the extraneous feature which the appeal to transmissibility of force introduces. Logically the best form of proof seems to be one on the lines adopted by Laplace; but the fact that he uses a differential equation to determine the law of composition may be regarded as a disadvantage. Mr. W. E. Johnson's proof, published in *Nature*, vol. XLI., p. 153, is one of the same class. It is a simple geometrical proof, and is in all respects admirable. It puts the proposition on its narrowest logical basis. All that has to be assumed is that the directed quantities, which are to be combined, have an unique resultant,

whose direction and ratio to one of them depends only on the angle between them and the ratio of their magnitudes.

The procedure to be followed practically in elementary teaching is a matter as to the details of which it would be idle to dictate. But, on the whole, it seems good to isolate any group of facts capable of isolation, and to encourage any ramifications of treatment that are available, rather than follow only one narrow path of deduction. At Winchester, thirty years ago, Mr. George Richardson, when lecturing on Mechanics, began with the principle of virtual work, as exhibited by elementary examples in statics. The present writer can testify to the fact that this procedure had the merit of arousing immediate interest in the subject.

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REVIEWS.

Leçons sur la Théorie des Formes et la Géométrie Analytique Supérieure, à l'usage des Etudiants des Facultés des Sciences. Par H. ANDOYER. Tome I. pp. vi., 508. Paris (Gauthier Villars), 1900.

It is a surprising fact that the projective invariant theory should have remained, till the very end of the century, unrepresented by any systematic treatise emanating from a French source. Quite early after its inception the study was benefited by the attention of French mathematicians; Hermite's striking contributions to it followed close on the original investigations of Boole, Cayley and Sylvester. Moreover French refinement in Analysis, and French leadership in Projective Geometry, more especially as developed from a few principles which are in effect the counterparts of facts of algebraical invariancy, led to the expectation that French Analytical Geometers would take the lead in expounding the algebraical theory as the key to the geometrical. But isolated memoirs, and partial consideration incidentally given in *Cours d'Analyse*, etc., have remained all that French authorship has provided.* Two years ago M. Andoyer put an end to the anomaly by the publication, for the use of candidates for "Agrégation," of a concise but comprehensive course of introductory lectures, *Léçons eléméntaires sur la Théorie des Formes et ses Applications Géométriques* (Paris, 1898, lithographed), and the promise of an extended manual of which the first volume is now before us. Examinations and authorship act beneficially on one another in France as here.

We have had long to wait; but ought now to be more than satisfied. M. Andoyer's aim is to present didactically the theory of Forms, and develop its application to a perfectly general geometry; and he is singularly successful. His plan is marked by great breadth of view. Possibly he is too chary of space when general lessons could be enforced and illustrated by particularisation. To be at once precise and general

* Faà de Bruno's *Formes Binaires* (Turin, 1876) is credited to Italy.

is throughout his effort. His expression is both forcible and abbreviated, and his notation is condensed. Considering the extreme compression it is truly remarkable how clearly his arguments are stated. The volume has had to be a bulky one, and it has evidently been always in the author's mind to allow no diffuseness or repetition to further enlarge it. Constant and painstaking collaboration is demanded from the reader by "il est clair," "facile de démontrer," "un calcul direct donne," and similar shortenings of demonstration. At times there is a suggestion of humour in the strife after brevity; as for instance when a short argument ends with a formula and the words "d'où une proposition facile à énoncer." We commend to our problem makers the original idea of sketching a solution and asking for the question.

Somewhat less than a third of the present volume is headed *La Géométrie Binaire*, and the remaining two thirds *La Géométrie Ternaire*. The second volume is to deal with *La Géométrie Quaternaire*. The headings call attention to—perhaps in the first section unduly emphasise—the prominence intended to be given to interpretation as distinct from purely Algebraical investigation. In keeping with such an intention, the author has excluded from his work what he calls the Arithmetical part of the Theory of Forms, as being of comparatively little importance with a view to geometry. There is no examination of general gradients or forms of a given type in order to extract information as to the invariants, etc., which belong to that type; and no investigation of irreducible systems of concomitants. The systems for forms of low orders (or *degrees* in M. Andoyer's terminology: he uses the term *order* in quite a new sense) are indeed given, but without devotion of space to the consideration of why they are complete systems. Thus our natural curiosity as to whether the "German" or the "English" method of investigation would be primarily adopted by M. Andoyer is set at rest in a somewhat unexpected manner. He eschews both. The umbral symbolism of Aronhold, Clebsch and Gordan is entirely absent from the work. So also is the combinatorial analysis of Cayley, Sylvester and MacMahon. No brevity consistent with intelligibility could bring within the limits of a single volume a satisfactory discussion of syzygetic theory and an equally satisfactory exhibition of the application of principles of invariancy to geometry; and we think that M. Andoyer has exercised a wise discretion in his omission.

That linear transformations form a continuous group is the basis on which the whole work rests, and Lie's doctrine of infinitesimal transformation is given the controlling position which it must henceforth retain in invariant theories. It may well be that Lie's inspiration was in part derived from previous knowledge, incompletely systematised, as to annihilators of invariants; but he repaid his debt a hundred-fold.

M. Andoyer has too wide views of his subject, and is too rigidly economical of his space, to first consider a single quantic or a single set of variables. He launches at once into the consideration of any number of sets (pairs or triads) of variables belonging to two different species—we regret that he can nowhere use the familiar terms *cogredient* and *contragredient*—and any number of forms in all or some of those

sets of variables. Neither can he find room to use *covariant, contravariant*, etc., as special terms. He once mentions the words (p. 27) but merely states the senses in which it is customary to employ them. Throughout the work *invariant* is his all-embracing word for every form, in the coefficients and sets of variables comprised in a system, which obeys the law of invariancy. It is impossible nevertheless for him absolutely to do without a term for *invariants* in the restricted sense. They figure from time to time as "invariants proprement dits." As, where they begin to do so, there is no reference to p. 27, and as on p. 27, the word "proprement" does not in fact occur, their introduction is at first not free from obscurity.

The systems of forms to which, after general investigations, special attention is devoted are, in the binary domain, linear and quadratic systems, and the cubic, quartic, quintic, bilinear, lineo-quadratic and doubly quadratic forms, and, in the ternary domain, linear quadratic and bilinear systems, and the cubic trilinear and quartic forms. The geometry of what we should call ranges and pencils is dwelt upon in connection with binary forms, and the study of what, to be narrow, we may describe as the theory of plane curves, receives prolonged attention in the second part of the volume.

But in geometrical applications M. Andoyer has had to face a considerable difficulty of nomenclature. He is not conservative, as we have seen, and he would waste space, or so he believes, by re-statement in the language of envelopes, for instance, of theorems once given in the language of loci, let us say. He has preferred to introduce a language of his own which shall be strictly impartial, and at once serve all purposes. He must at all costs be general. He will not be content to mean or seem to mean by Analytical Geometry merely the application of Analysis to actual geometry, much less to a one-sided actual geometry. Higher Analytical Geometry is to him the creation, so to speak, of Analysis. Algebras and Geometries have not a one to one correspondence. One geometry, the actual geometry, has led to one Algebra as very closely expressing it. The one Algebra leads in return not to the one old geometry only, but to many geometries. What is to be studied then is the quasi-geometrical expression of what arises in the most general conceivable *continuum* which is doubly filled by elements of two species, elements assigned as corresponding under consistent laws to the sets of values of certain algebraical variables of two species which we should describe as contragredient. Actual geometry is included in, but of course does not exhaust the conception. To be general then our author will speak, not of a curve or an envelope, but of a "série" of the first or second species. He will say, not inflexion or cusp, but "élément inflexionnel" of the one species or the other, and so on.

Instructive chapters are devoted to "metrical" geometry of one and two dimensions—metrical in a general sense of course. The generalized ideas of distance and rectangularity are well expressed. A quadratic "série" bitangent to the absolute is a "cercle," but its reduced tangential equivalent is a "cycle." The consideration of "cercles" leads to an anticipation of quaternary geometry. Particularisation as to the metrics

of actual geometry is, in accordance with the spirit of the work, only mentioned to be put aside—"il est inutile d'insister sur ce point évident."

Reference to authorities is rare in the book—indeed almost absent. The author pleads the didactic character of the work as inconsistent with it. We can well understand this. He who would write an effective didactic treatise must first saturate himself with his subject, and then, putting authority aside, express the ideas which he has made his own. The right acknowledgement in the right place thus becomes often more than difficult. Without doubt this has been M. Andoyer's method. He promises a bibliography in the volume still to appear.

We hope he will at the same time give us an alphabetical index to the whole work. A good book gains much from such an aid to facility of reference. An author can provide it with an amount of labour which, though tedious, is as nothing to what he has devoted to the preparation of his work. It has become quite the rule for English authors to thus help their readers. Unfortunately it is still rare for French ones to do so. As M. Andoyer has not yet inaugurated a better practice, we commend to him the example, set in Lie-Engel's *Continuierliche Transformations-Gruppen*, of appending a full index to the final volume.

To the appearance of this final volume we shall look forward with sanguine expectations. E. B. ELLIOTT.

The Fifth and Sixth Books of Euclid, arranged and explained by Professor M. J. M. HILL, M.A., D.Sc., F.R.S. (Cambridge: University Press; pp. xx., 143.)

This edition of Euclid V. and VI. deserves a welcome from all who desire the improvement of geometrical teaching. Although in the main depending upon Euclid's definition of proportion, the book differs from Euclid in leading gradually up to this idea of proportion, in treating the Sixth Book concurrently with the Fifth, and in methodically grouping the propositions of the Sixth Book. It is sincerely to be hoped that this plan of combining the two books will be followed in future by text-books covering the same ground, as it is a decided gain to apply each principle to some practical example, as soon as it has been established.

In the preface, Prof. Hill says that the object of the book is to remove the chief difficulties of those who wish to understand Euclid's Sixth Book. For this purpose he employs scales of magnitudes, as described later, and postpones the whole question of ratio, until a good conception of Euclid's test of equal ratios has been gained under another form.

The book is divided into ten sections.

Section I. gives, with geometrical illustrations, general theorems on multiples, some of which, being obvious when stated (*e.g.* if $A > B$, $rA > rB$,), might be given without proof.

Scales are introduced in Art. 28. Multiples of any magnitude A may be represented graphically by means of a linear scale, familiar to most people in the foot-rule or inch-tape. It should be remembered that the scale represents multiples of *any* kind of magnitude whatever.

Arts. 31-39 show how any two magnitudes A and B, of the same kind, may be compared by means of scales of their multiples. The scales of A and B are set side by side, and the comparison effected by a table of their successive multiples in order of magnitude.

For example, taking comparative scales of inches I, and centimetres C,

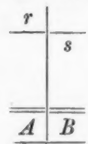
their multiples, arranged in order of magnitude, are

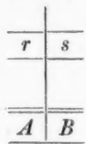
$$C,\ 2C,\ I,\ 3C,\ 4C,\ 5C,\ 2I,\ 6C,\ 7C,\ 3I,\ \text{etc.} \ldots\ldots\ldots\ldots \text{(M)}$$

Thus, as the scale extends we get a more and more accurate idea of the relative magnitudes of I and C. The method, it should be remembered, applies to *any* two magnitudes of the same kind.

The scale thus formed of successive multiples of two magnitudes A and B of the same kind, is called by Prof. Hill 'the relative multiple scale of A, B'; or, briefly, 'the scale of A, B'; and is denoted by $[A, B]$.

The symbol $\simeq$ denotes 'is the same as,' and the diagrams

(i) 

(ii) 

(iii) 

denote, r and s being integers,

$$\text{(i) } rA > sB, \quad \text{(ii) } rA = sB, \quad \text{(iii) } rA < sB.$$

Now if (Art. 35) the scale of A, B is the same as the scale of C, D, the order of multiples of A, B (as in the line marked M above) must be the same as the order of multiples of C, D. That is, if

$$[A, B] \simeq [C, D];$$

then, r and s being integers,

$$\begin{array}{lll} \text{whenever} & rA > sB,\ rC > sD, & \text{and conversely;} \\ \quad ,, & rA = sB,\ rC = sD, & \quad ,, \\ \quad ,, & rA < sB,\ rC < sD, & \quad ,, \end{array}$$

Thus there are six sets of conditions satisfied when two scales are the same. These conditions are not, however, all independent.

In Art. 36, Prop. VIII., it is shown that if for the scales of A, B and C, D the three sets of conditions written in full above are given, the converses are true, *i.e.* if

$$\text{whenever } rA \gtreqless sB,\ rC \gtreqless sD,$$

then also $$\text{whenever } rC \gtreqless sD,\ rA \gtreqless sB;$$

and hence (without further proof),

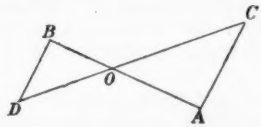
the scale of A, B is the same as that of C, D.

The proof would be clearer here, if, after 'hence,' were inserted 'the order of multiples of A, B is the same as that of C, D, and therefore,' etc.

We are thus in Section I. gradually led up to the notion of the sameness of scales, and thus of the sameness of the relative magnitude of two pairs of magnitudes, A, B and C, D, without any definition of ratio, and it is clear that the above conditions for the sameness of scales are the same as those in Eucl. V., Def. 5, for the sameness of ratios. The graphic method employed is of great assistance towards a complete understanding of Euclid's definition, and when once grasped is much simpler than it appears to be at first.

Section II. proves some theorems of Eucl. V., including V. 11, 'Scales which are the same as the same scale are the same as each other'; and shows from first principles (instead of from VI. 1, which is proved in a later section) that two lines are similarly divided by a system of parallels. It gives also (Art. 57) the construction for a fourth proportional to three given lines.

It is a pity that the traditional construction is here used, instead of the following



from which it is plain that *any* three of the four terms OA, OB, OC, OD, being given, the fourth can be determined, by drawing the necessary parallel, so that (from Art. 56)

$$OA : OB = OC : OD.$$

Section III. gives Euclid's Definitions of Ratio and Equal Ratios, and propositions are henceforth doubly stated in terms of scales and ratios.

An important proposition is assumed in Art. 64, which Prof. Hill calls 'The Fundamental Proposition in the Theory of Relative Multiple Scales,' viz.:

'If there be two magnitudes A, B of the same kind, and any other magnitude D; then there exists one and only one magnitude C such that the scale of C, D is the same as that of A, B.'

This will be referred to later; its assumption is necessary because the definition of ratio is incomplete.

The remaining sections complete most of Bk. V. and all Bk. VI., with A, B, C, D and a few propositions and definitions of modern geometry; similar figures, areas, etc., being dealt with in different sections. This is one of the distinctly good features of the book. Many of the proofs are simpler than the traditional ones, and necessary completions are added to proofs often left incomplete. Thus in compounding ratios it is proved that if $a : b = l : m$ and $c : d = m : n$, the ratio $l : n$ is the same whatever quantity we choose arbitrarily as l. The proof given that parallelograms have the ratio compounded of those of their sides, is very simple; and very properly, duplicate ratio is treated as the result of compounding a ratio with itself.

As Prof. Hill invites suggestions on the book, it seems to me a pity that he has not gone further in sacrificing the Fifth Book to the Sixth. Eighteen Fifth book propositions are proved, but of these only

7, 9, 11, 12, 14, 15, 16, 18, 22, 24

are quoted in Book VI., and of these 15, 18, and 24 are quite unnecessary. It would be worth while to omit unnecessary propositions, in order to make room for more work on modern geometry.

And it is disappointing that one of the root difficulties, the lack of an exact definition of ratio, is left untouched. It is proved in Art. 44 that the ratio of two whole numbers $r : s$ is the same as that of the fraction r/s to unity, and then *stated* that 'the fraction r/s is taken to be the measure of the ratio $r : s$.' But why? It is *not* proved that the fraction r/s is equal to the ratio $r : s$, but only that the *ratio* $r/s : 1 =$ the ratio $r : s$, a very different proposition. And as a matter of fact, either of the fractions $\frac{r+s}{s}$, $\frac{r}{r+s}$ will satisfy all Euclid's definitions quite as well as r/s, and therefore we cannot *deduce* the *value* of the ratio $r : s$ from these definitions. Moreover the value r/s is assumed in Trigonometry and Conics, on the ground that ratio has been completely dealt with in Euclid, so that we move in a vicious circle.

There is a further difficulty that, in dealing with compound ratio, it has to be *assumed* that a fourth proportional always exists to any three magnitudes a, b, l*; this is only *proved* in Euclid when a, b, l are lines (or, by extension, areas). Thus at this point Euclid's method applies to geometry only, and *ceases to be general.*

These seem to me to be very strong reasons for making a distinct change from Euclid in the definition of ratio, which is suggested as follows:

'The ratio of a magnitude A to another B of the same kind is the measure of A in terms of B as unit.'

This gives at once the fraction A/B (A and B being measured in terms of a common unit) as the value of the ratio, and gets rid of V. 7, 9, 11, 14, which become axiomatic; and leaves only V. 16 (alternando), 22 (ex aequali), and 12 (which includes 15), viz.: 'The ratio of the sum of the antecedents of a number of equal ratios to the sum of the consequents is the same as any of the ratios.' I would suggest that these three can be readily proved for lines or areas geometrically. This definition also makes it possible to at once determine a fourth proportional to three given magnitudes, and the difficulty of incommensurable ratios can be evaded by proving Euclid's test of proportion as a theorem.

The book is beautifully printed, the proofs are very clear, and numerous carefully chosen examples are given bearing upon both processes and results. It will, I trust, raise the whole question of the treatment of ratio, and it ought to be read and studied by all teachers and students of geometry. E. Budden, M.A., B.Sc.

* Prof. Hill's Fundamental Proposition.

[We append Prof. Hill's rejoinder to the above acute *critique*. The importance of the subject to those teachers who realise the difficulties of Books V. and VI. seems to justify this departure from our usual custom.—EDITOR.]

By the courtesy of the Editor of the *Mathematical Gazette* an advance copy of the above review has been sent to me with an intimation that I might make some remarks upon it. Of this opportunity I gladly avail myself.

(1) It may prevent misconception to say that the Fundamental Proposition in the Theory of Scales is proved in my article on the Fifth Book of Euclid's Elements in the *Cambridge Philosophical Transactions*, Vol. XVI., Part IV., in a perfectly *general* manner. It is far too difficult to be included in an elementary text-book. But it should be noticed that all that is required for a rigorous treatment of the Sixth Book of Euclid is a proof of this proposition for the *special* case where the magnitudes concerned are segments of straight lines (which is given in Euc. vi. 12), so that there is no lacuna left in the argument for those who do not desire to go beyond the Sixth Book.

(2) The reference to Prop. viii. does not include the second part of that proposition, which with Prop. vii. render it possible to dispense with the 7th definition of the Fifth Book, and thus get rid of the indirectness of Euclid's line of argument, as explained in the preface. This is the principal improvement in the treatment of the whole subject in the book; and it is this, more than anything else, which brings the argument within the grasp of beginners.

(3) The suggestion that ratio should be defined thus: 'The ratio of a magnitude A to another B of the same kind is the measure of A in terms of B as unit' is open to very serious objection. This definition, though very simple in appearance, is very complex in reality.

When the two magnitudes are commensurable, say $A=rG$, and $B=sG$, it leads readily to the rational fraction $\frac{r}{s}$ as the measure of the ratio of A to B.

But if the magnitudes are incommensurable, then it raises the whole question of incommensurables, which I have not attempted to deal with, because I regard it as wholly beyond the beginner. I have, however, so arranged the argument that when the beginner reaches the theory of incommensurables he does not find he has anything to unlearn.

(4) The reviewer says, 'it is disappointing that one of the root difficulties, the lack of an exact definition of ratio, is left untouched.'

I do not know that any better definition of ratio has been given than that of De Morgan, viz. 'relative magnitude.' But this is verbal only, and does not help to measure ratio. I have explained in the Article referred to how to measure ratio, but this necessarily follows the proof of the Fundamental Proposition in the Theory of Scales. When that has been effected it seems to me that to seek for a further definition of ratio is quite useless, and is in any case a matter of words only.

(5) The suggestion that if the definition that 'the ratio of a magnitude A to another B of the same kind is the measure of A in terms of B as unit' be adopted, then Euc. v. 7, 9, 11, 14 would become axiomatic, only leads to the introduction of the inexact arithmetical theory, applicable to commensurable ratios only.

(6) The suggestion that Euc. v. 12, 16, 22 may 'be readily proved for lines or areas geometrically,' and that the definition 'also makes it possible to at once determine a fourth proportional to three given magnitudes, and the difficulty of incommensurable ratios can be evaded by proving Euclid's test of proportion as a theorem,' can only be dealt with when set forth in full.

(7) The answer to the enquiry why I take r/s as the measure of $r:s$ is that it is convenient. A rigorous argument could still be conducted if $\frac{r+s}{s}$ were taken to measure the ratio of $r:s$, provided this were adhered to throughout. Only this would entail more trouble, just as the addition of 10 to the ordinary logarithm to form the tabular logarithm is (as one of my students informed me) a specially constructed device for the annoyance of the beginner.

(8) The statement that Euc. v. 15, 18 and 24 are quite unnecessary is one with which I can only express my disagreement. It may be possible to prove the propositions in the Sixth Book without them, but they are valuable for other purposes in Geometry. The propositions in the Fifth Book which are not

required for the Sixth Book have been collected together in the Tenth and Eleventh Sections, so that they are clearly marked off from that which must be read to understand the Sixth Book; but the beginner will find them more and more useful as he goes deeper into the study of Geometry. M. J. M. Hill.

MATHEMATICAL NOTES.

88. [K. 20. d.] *Geometrical proofs:*

$$\text{(1)}\quad \sqrt{1+\sin A}+\sqrt{1-\sin A}=2\cos\frac{A}{2}\qquad (A<90^\circ).$$

$$\text{(2)}\quad \sqrt{\frac{1-\cos A}{2}}=\sin\frac{A}{2}\qquad (A<90^\circ).$$

$$\text{(3)}\quad \frac{-1+\sqrt{1+\tan^2 A}}{\tan A}=\tan\frac{A}{2}\qquad (A<90^\circ).$$

Let P be any point on a semicircle APB, radius unity, centre O, diameter AB. Let $P\hat{O}B$ be A. Draw PN perpendicular to AB.

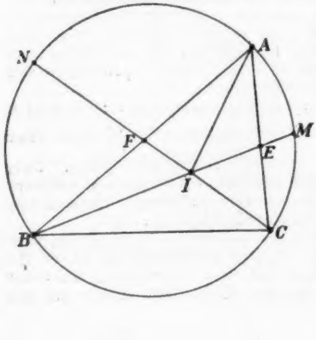
$$\begin{aligned}\text{(1)}\quad \sqrt{1+\sin A}+\sqrt{1-\sin A}&=\sqrt{1+PN}+\sqrt{1-PN}\\ &=\sqrt{1+\sqrt{AN\,.\,NB}}+\sqrt{1-\sqrt{AN\,.\,NB}}\\ &=\sqrt{\frac{AN+NB+2\sqrt{AN\,.\,NB}}{2}}+\sqrt{\frac{AN+NB-2\sqrt{AN\,.\,NB}}{2}}\\ &=2\sqrt{\frac{AN}{2}}=2\sqrt{\frac{AN}{AB}}=2\,.\,\frac{AN}{AP}=2\cos\frac{A}{2}.\end{aligned}$$

$$\text{(2)}\quad \sqrt{\frac{1-\cos A}{2}}=\sqrt{\frac{1-ON}{2}}=\sqrt{\frac{NB}{AB}}=\frac{NP}{PB}=\sin\frac{A}{2}.$$

$$\begin{aligned}\text{(3)}\quad \frac{-1+\sqrt{1+\tan^2 A}}{\tan A}&=\frac{-1+\sqrt{1+\dfrac{PN^2}{ON^2}}}{\dfrac{PN}{ON}}\\ &=\frac{-ON+\sqrt{ON^2+PN^2}}{PN}=\frac{-ON+1}{PN}=\frac{NB}{PN}=\tan\frac{A}{2}.\end{aligned}$$

J. V. Thomas.

89. [K. 1. b.] *The equality of internal bisectors.*



Let ABC be a triangle, and let the internal bisectors of the angles at B, C meet the opposite sides in E, F and the circumcircle in M, N respectively.

Given $BE=CF$, to prove $\hat{B}=\hat{C}$.

Since $BE\,.\,BM=AB\,.\,BC$,

$CF\,.\,CN=AC\,.\,BC$;

$\therefore\ BM : CN=AB : AC$.

and $CN\,.\,AB-BM\,.\,AC=0$,

or $AN(AC+BC)-AM(BC+AB)=0$,

or

$BC(AN-AM)+AC\,.\,AN-AB\,.\,AM=0.$

Now $AN >=$ or $< AM$ according as $\hat{C} >=$ or $< \hat{B}$,

i.e., as $$\frac{\hat{A}}{2}+\frac{\hat{C}}{2} >=< \frac{\hat{A}}{2}+\frac{\hat{B}}{2},$$

or as perp. from A on $CN >=<$ perp. from A on BM,

or as $$AC \,.\, AN >=< AB \,.\, AM.$$

Hence $$AN = AM \text{ and } \hat{C} = \hat{B}.$$

R. F. DAVIS.

90. [K. 20. F.] In any spherical triangle ABC produce BC to D so that CD is a quadrant. Then from the fundamental formula

$$-\cos c \sin a + \sin c \cos a \cos B = \cos AD = -\sin b \cos C,$$

or, rearranging and dividing by $\sin C$,

$$\sin a \cot c = \sin B \cot C + \cos a \cos B.$$

W. E. HARTLEY.

91. [L[1]. 7. d.] *If one focus of a conic inscribed in a triangle lies on the line joining the circum- and the ortho-centres of the triangle, the locus of the other focus will be an equilateral hyperbola circumscribing the triangle and passing through its circum-centre.*

Let $$l\alpha + m\beta + n\gamma = 0$$

be the straight line, then since

$$\cos A : \cos B : \cos C,$$
$$\sec A : \sec B : \sec C$$

are the coordinates of the points through which it passes, we have

$$l \cos A + m \cos B + n \cos C = 0, \quad \text{.........(1)}$$
$$l \sec A + m \sec B + n \sec C = 0, \quad \text{.........(2)}$$

but if $\alpha_1 : \beta_1 : \gamma_1$ is one focus,

$$\frac{1}{\alpha_1} : \frac{1}{\beta_1} : \frac{1}{\gamma_1}$$

denotes the coordinates of the other, therefore the locus of the other focus is

$$\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=0. \quad \text{.........(3)}$$

The condition (1) makes (3) an equilateral hyperbola, and (2) is the condition that it should pass through the circum-centre.

J. M. DYER.

PROBLEMS.

[*Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by Proposers.*]

390. [K. 13. a.] If $A_1, \ldots A_6$ be any six points ranged in this order on the circumference of a circle, and O any point in space,

$$OA_1^2/\Pi(A_1A_s) + OA_3^2/\Pi(A_3A_s) + OA_5^2/\Pi(A_5A_s)$$
$$= OA_2^2/\Pi(A_2A_s) + OA_4^2/\Pi(A_4A_s) + OA_6^2/\Pi(A_6A_s),$$

where $\Pi(A_1A_s) = A_1A_2 \,.\, A_1A_3 \,.\, A_1A_4 \,.\, A_1A_5 \,.\, A_1A_6$, etc.

SIR ROBERT BALL

391. [K. 20. e.] In any triangle prove that

$$(a+b-2c)^2\sec^2\frac{C}{2}+(a-b)^2\operatorname{cosec}^2\frac{C}{2}=(b+c-2a)^2\sec^2\frac{A}{2}+(b-c)^2\operatorname{cosec}^2\frac{A}{2}$$
$$=(c+a-2b)^2\sec^2\frac{B}{2}+(c-a)^2\operatorname{cosec}^2\frac{B}{2}$$

and interpret geometrically. E. N. Barisien.

392. [L[1]. 4. c.] Two parallel tangents TR, PQ, are drawn to a conic; O is the mid-point of TP, a tangent perpendicular to the former two, and RSQ is perpendicular to SO. If W be the fourth harmonic to R, S, Q, then is WO a tangent to the conic. C. V. Durell.

393. [M_1. 2. a.] Salmon, *Higher Plane Curves*, p. 148, footnote: "It is easy to see that we may have nine real points lying by threes on ten straight lines but not in a greater number of lines."

Given four points A, B, C, D, not all in a straight line, give a geometrical construction for the remaining five points to complete a set satisfying the above conditions. C. S. Jackson.

394. [P. 2. a.] A variable conic S touches a given line at a given point P and is reciprocated with respect to any fixed origin O into a conic Σ touching a given line at a given point π: prove that the product of the radii of curvature of S at P and Σ at π is constant for all pairs of conics such as S, Σ. A. P. Thompson.

395. [K. 5. a.[If ABC, $A'B'C'$ be triangles inversely similar, and O their double point, then OA', OB', OC' meet BC, CA, AB respectively in three collinear points. What is the corresponding property for directly similar triangles? C. E. Youngman.

396. [I. 25. b.] Required to find a triangular number which can be expressed as the sum of two other triangular numbers in five different ways. [*N.B.*—There is only one solution in numbers less than 5000.]

W. Allen Whitworth.

397. [A. 3. 1; K. 20. c. α.] Find the greatest root of $x=100\sin x$. (C.)

398. [K. 10. e.] The square of the tangent from the origin to the circle circumscribing the triangle formed by the lines

$$lx+my=1, \quad mx+ny=1, \quad nx+ly=1$$

is $$\Sigma(m^2+n^2)(mn-l^2)/\Sigma mn(m^2+n^2)(mn-l^2). \qquad \text{(C.)}$$

399. [L[1]. 17. e. 19. d.] Two parabolas have a common focus; from any point on a common tangent are drawn the other tangents to the parabolas: shew that another parabola with the same focus can be drawn touching the last two lines and the join of their points of contact. (C.)

SOLUTIONS.

Unsolved Questions.—171, 275, 279, 283, 326, 336-8, 341, 349, 356, 369, 370, 373, 375-9, 380-4, 387-9.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn **to a small scale** on a separate sheet. Solutions will be published as space is available.

19. [K. 12. a.] *Describe a circle passing through a given point A, and subtending given angles β, γ at two other given points B, C.* Prof. Neuberg.

Solution by R. F. Davis, E. M. Langley.

Suppose the centre O of the $\odot$ to be found,

$$OC\sin\frac{\gamma}{2}=r=OB\sin\frac{\beta}{2}=OA;$$

$\therefore$ O lies on two fixed circles, and therefore is one of two fixed points.

[The solution would apply equally well to the case of a $\odot$ subtending angles α, β, γ at A, B, C respectively.]

21. [K. 4. c.] *On the base BC of a triangle ABC is described an isosceles triangle BCD. BD, CD meet AC, AB respectively in E, F. Show that the envelope of EF is a conic inscribed to the triangle ABC.* PROF. NEUBERG.

Solution by R. TUCKER.

If the base angles of DBC are θ, the equation to EF will be found to be

$$[c(a-x)-ay]\sin\theta\sin(B-\theta)=b\sin(\theta-C)[x\sin\theta-y\sin(B-\theta)];$$

or $$t^2[c\cos B(a-x-y\cos B)+bx\cos C]-act\sin B+cy\sin^2 B=0$$

where $$t=\tan\theta.$$

Hence EF envelopes the hyperbola, whose equation is

$$a^2c=4y[ac\cos B+x(b\cos C-c\cos B)-cy\cos^2 B],$$

which has BC as asymptote, and touches BA at $y=\frac{a}{2}\sec B$.

Solution by ANON.

Let D be any point within the triangle, BD, CD cutting AC, AB in EF respectively. If $D(\alpha_1, \beta_1, \gamma_1)$ lie on a straight line $\Sigma l\alpha=0$; EF is $-\frac{\alpha}{\alpha_1}+\frac{\beta}{\beta_1}+\frac{\gamma}{\gamma_1}=0$, and touches the in-conic $\sqrt{-l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$.

Or again, since locus of D is a straight line, $B\{D\}=C\{D\}$; $\therefore$ $\{E\}=\{F,\}$ and AC, AB are homographically divided in E, F. $\therefore$ EF envelopes a conic inscribed in the triangle. [This is the converse of:—$BFEC$ is a quadrilateral circumscribed to a conic; its diagonal triangle is self-conjugate, and if one diagonal pass through a fixed point A, its pole D describes the polar of A.] No. 21 is a particular case, for D is on the perpendicular bisector of BC. The point of contact of BC and the envelope is at infinity; $\therefore$ BC is an asymptote.

23. [K. 20. f.] *In a spherical triangle* $\cos a=\frac{3}{5}$, $\cos b=\frac{5}{13}$, $\cos C=60°$. *Show* $a=c$. W. GALLATLY.

Solution.

$$\cos c=\cos a\cos b+\sin a\sin b\cos C=\frac{3}{5}\cdot\frac{5}{13}+\frac{4}{5}\cdot\frac{12}{13}\cdot\frac{1}{2}=\frac{15+24}{65}=\frac{3}{5}=\cos a.$$

$$\therefore\ a=c.$$

24. [K. 1. c.] *If two sides of a triangle are given in position and its perimeter in magnitude, show that the third side envelopes a fixed circle, without using the centre of the circle or the equality of tangents from the same point.*

E. M. LANGLEY.

Solution by PROPOSER.

Let AB, AC be given in position, BC the variable side. Take L, M, N on BC, CA, AB, so that $AB+BL=AC+CL=AM=AN=$semi-perimeter.

$BL=BN$; $B\hat{L}N=\frac{1}{2}B$.

Similarly $C\hat{M}L=\frac{1}{2}C$. Then $B\hat{L}N+C\hat{L}M=\frac{1}{2}(B+C)=A\hat{M}N$. $\therefore$ $M\hat{L}N$ is constant.

$\therefore$ L lies on a fixed circle. Also $\because$ $B\hat{L}N=A\hat{M}N-C\hat{M}L=L\hat{M}N$; $\therefore$ BC touches the circle at L.

25. [R. 4. a.] *Give a statical proof that* $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\ldots$ ad inf. $=\frac{1}{3}$.

E. M. LANGLEY.

Solution by PROPOSER.

Take a straight line AB of unit length; consider the C.G., G, of weight $2w$ at A, w at B. $AG=\frac{1}{3}AB$. Bisect AB in G_1; AG_1 in G_2; G_1G_2 in G_3, and so on. The system is equivalent to $2w$ at G_1 and w at A, which is equivalent to $2w$ at G_2, and w at G_1, and so on. Also $G_rG_{r-1}=\frac{1}{2^r}\cdot AB$, and hence decreases without limit. Hence AG is the limit of AG_r, *i.e.* is the limit of $AG_1+G_1G_2+G_2G_3+\ldots$ regard being had to sign.

$$\therefore\ \tfrac{1}{3}=\text{limit of } \tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{8}-\ldots \text{ ad inf.}$$

27. [I. 1.] *Given* $\log 1{\cdot}3712={\cdot}1371008$, $\mu={\cdot}4343$, *find* x *to six significant figures, where* $10\log x=x$.

E. M. LANGLEY.

Solution by PROPOSER.

Let
$$x=1{\cdot}3712+h,$$
$$10\log(1{\cdot}3712+h)=1{\cdot}3712+h,$$
$$10\log 1{\cdot}3712=1{\cdot}371008,$$
$$10\log\left(1+\frac{h}{1{\cdot}3712}\right)={\cdot}000192+h=\frac{4{\cdot}343h}{1{\cdot}3712}\text{ approx.};$$
$$\therefore\ h={\cdot}00009,$$
$$x=1{\cdot}37129.$$

Solution by G. HEPPEL.

Let $x=u+y$ where $u=1{\cdot}3712$, $v={\cdot}1371008$.

$$u+y=10\log(u+y)=10\log u+10\log(1+y/u)=10v+10\mu y/u\text{ approx.},$$
$$y=(u-10v)u/(10\mu-u)={\cdot}0000885,$$
$$x=1{\cdot}3712885.$$

29. [K. 9. b.] *A blindfolded man continually walks straight a constant distance and then wheels left through a constant angle. Prove by elementary geometry, that if the angle be a submultiple of four right angles, he will return to the starting point.*

R. W. GENESE.

Solution by E. M. LANGLEY.

If AB, BC, CD be three successive elements of the broken path, and if the bisectors of $A\hat{B}C$, $B\hat{C}D$ intersect at O, the triangles AOB, BOC, COD are clearly congruent; hence AB, BC, CD ... are chords of the same circle, and $A\hat{O}B=\pi-A\hat{B}C=$constant angle. If this constant angle be a submultiple of 2π, the chords are the sides of a regular in-polygon of the circle.

33. [K. 20. c.] *Solve* $\dfrac{x+\sqrt{1-x^2}}{1+x+\sqrt{1-x^2}}=\dfrac{x-\sqrt{1-x^2}}{1-x+\sqrt{1-x^2}}$ *by Trigonometry.*

E. M. LANGLEY.

Solution by PROPOSER.

$$x=\cos\theta \text{ gives } \frac{\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta}=\frac{\cos\theta-\sin\theta}{1-\cos\theta+\sin\theta} \text{ whence } \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\tan\theta.$$

$$\therefore\ 3\theta=2n\pi+\frac{\pi}{2}\text{ and }\cos\theta=0\text{ or }\pm\sqrt{3}/2.$$

[With $x=0$ we must take $\sqrt{1-x^2}=-1$; with $x=\pm\frac{1}{2}\sqrt{3}$, $\sqrt{1-x^2}=+\frac{1}{2}$.]

34. [A. 2. a.] *The p^t term of an A.P. is k, the q^{th} term is l; find the $(p+q)^{th}$ term, and account geometrically for the result.* E. M. LANGLEY.

Solution by PROPOSER.

Let the terms of the A.P. be represented by abscissae measured along a fixed axis OX from a fixed origin O.

Let $OP=k$. $OQ=l$. $OR=(p+q)^{th}$ term. Then PR, QR contain the common difference q times and p times respectively; *i.e.* R divides PQ externally as $q:p$. $\therefore\ OR=(ql-pk)/(q-p)$.

35. [A. 2. a.] *In a certain examination every candidate must take either Latin, or Mathematics, or both. Of the candidates,* 79·4 % *took Latin, and* 89·6 % *took Mathematics. If there were* 1500 *candidates altogether, what are the greatest and least numbers that could have taken both Latin and Mathematics?* St. Andrews Loc. Ex., 1892.

Solution.

Suppose x took Latin alone, y Mathematics alone, z both Latin and Mathematics.

Then $\dfrac{79{\cdot}4}{100}.1500=x+z$; $\dfrac{89{\cdot}6}{100}.1500=y+z$; $x+y+z=1500$;

whence $x+y+2z=\dfrac{169}{100}.1500$ $\therefore\ z=\dfrac{69}{100}.1500=1035$. $\therefore\ z=156$; $y=309$.

[The question would appear to be incorrectly stated. Perhaps for *must* we should read *may*; the two numbers would then be 1191 and 1035.]

41. [L¹. 17. e.] *If a parabola and an ellipse have a common latus rectum, the intercept on the major axis between the focus and a common tangent is bisected at a vertex of the ellipse.* P. J. HEAWOOD.

Solution by PROPOSER.

Y being the foot of the perpendicular from S on a tangent YT to an ellipse, so that AYA' is a right angle; and YN perpendicular to AA':

YT will also touch the parabola, focus S, vertex N, latus rectum $=4SN$.

If this = the latus rectum of the ellipse $=\dfrac{4AS.SA'}{AA'}$, we have

$$(AS-AN)AA'=AS(AA'-AS);\ \text{or}\ AS^2=AN.AA'=AY^2.$$

$$\therefore\ AS=AY=AT\ (SYT \text{ being a right angle}),$$

i.e. the vertex A of the ellipse bisects ST.

80. [A. 2. b.] *Prove that $\dfrac{x+a}{x^2+bx+c}$ will always lie between two fixed finite limits if $a^2+c^2>ab$ and $b^2<4c^2$; that there will be two limits between which it cannot lie if $a^2+c^2>ab$ and $b^2>4c^2$; and that it will be capable of all values if $a^2+c^2<ab$.*

Correction and Solution by A. LODGE; W. E. JEFFARES.

[c must replace c^2 in the above conditions. Also since

$$a^2+c-ab=\tfrac{1}{4}\{(2a-b)^2+4c-b^2\},$$

which is positive if $b^2<4c$, the sole condition in the first part is that $b^2<4c$; and further, as $4c>b^2$, c is positive.]

Let $2(x+a)=y$; $(2a-b)=m$ where m is positive or negative, and (1) .et $4c-b^2=n^2$.

Then $u=\dfrac{x+a}{x^2+bx+c}=\dfrac{2y}{(y-m)^2+n^2}$, which is positive or negative with y,

$$=2\Big/\left\{y+\frac{m^2+n^2}{y}-2m\right\}.$$

This is a maximum, for $y=\dfrac{m^2+n^2}{y}=\pm\sqrt{m^2+n^2}$.

$\therefore$ u lies between $\dfrac{1}{\pm\sqrt{m^2+n^2}-m}$, *i.e.* between $\dfrac{1}{b-2a\pm2\sqrt{a^2+c-ab}}$.

(2) Let $b^2-4c=n^2$, so that $\frac{1}{4}(m^2-n^2)=a^2+c-ab$.

Then $$u=\frac{2y}{(y-m)^2-n^2}, \text{ or } uy^2-2y(um+1)+u(m^2-n^2)=0.$$

$\therefore$ if y is real, $(um+1)^2-u^2(m^2-n^2)$ is positive.

This is always the case if m^2-n^2 is negative, *i.e.* if $a^2+c<ab$.

If $m^2>n^2$, the condition becomes $u+\dfrac{1}{m\pm\sqrt{m^2-n^2}}>0$.

$\therefore$ u cannot lie between the values $-\dfrac{1}{m\pm\sqrt{m^2-n^2}}$,

i.e. between $\dfrac{1}{b-2a\pm2\sqrt{a^2+c-ab}}$.

274. [D. 6. b. γ.] (Extension of 235.) *If*

$$\sum_{r=1}^{n-1} x^2_r-\operatorname{sech} a\sum_{r=1}^{n-2} x_r x_{r+1}=C, \quad\text{(A)}$$

prove that

$$(1)\ x^2_r\leqslant 2c\frac{\sinh(n-r)a\sinh ra\cosh a}{\sinh na\sinh a};$$

$$(2)\ \frac{c\cosh a}{\cos\frac{\pi}{n}+\cosh a}\leqslant\Sigma x^2_r\leqslant\frac{c\cosh a}{\cosh a-\cos\frac{\pi}{n}};$$

(3) *The maximum and minimum values of* $x_r x_{r+s}$ *are* $c\cosh a/(d_1\sinh a)$ *and* $c\cosh a/(d_2\sinh a)$, *where* d_1, d_2, *are the roots of*

$$(d\sinh sa+1)^2\sinh ra\sinh(n-r-s)a=\sinh(r+s)a\sinh(n-r)a.$$

(Corrected.) A. C. L. Wilkinson.

Solution by Proposer.

The equation $\Sigma x_r^2-\operatorname{sech} a\Sigma x_r x_{r+1}=C$ represents an ellipsoid in $n-1$ dimensions. For (cf. Burnside and Panton, *Theory of Equations*, § 195) we have

$$\Sigma x_r^2-\operatorname{sech} a\Sigma x_r x_{r+1}\equiv p_1X_1^2+\ \dots\ +p_{n-1}X_{n-1}^2$$

where $$p_r=\Delta_r/\Delta_{r-1},$$

Δ_r being the discriminant of the function

$$\sum_{\nu=1}^{\nu=r} x_\nu^2-\operatorname{sech} a\sum_{1}^{\nu-1} x_\nu x_{\nu+1},$$

and from the expression of Δ_r as a determinant the relation

$$\Delta_r=\Delta_{r-1}-\tfrac{1}{4}\operatorname{sech}^2 a\Delta_{r-2}$$

easily follows.

Hence we can obtain the expression of Δ_r, viz.

$$\Delta_r=\frac{\sinh(r+1)a}{\sinh a(2\cosh a)^r},$$

and is always positive when a is real. Thus $p_1\dots p_{n-1}$ are all positive, and the given equation of condition represents an ellipsoid, c being positive.

From this preliminary investigation it is obvious that true maxima and minima exist of x_r^2, Σx_r^2, and $x_r x_{r+s}$, and these maxima and minima may be found by differentiation and the use of undetermined multipliers.

(1) To make $x_r^2=V$ a maximum, subject to A, we have by differentiation the system:

$$\left.\begin{array}{r}2x_1-\operatorname{sech}\alpha\,.\,x_2=0\\ 2x_2-\operatorname{sech}\alpha(x_1+x_3)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ 2x_{r-1}-\operatorname{sech}\alpha(x_{r-2}+x_r)=0\end{array}\right\};\quad \begin{array}{r}-2kx_r+2x_r\\ -\operatorname{sech}\alpha\\ \times(x_{r-1}+x_{r+1})\\ =0\,;\end{array}\left\{\begin{array}{r}2x_{r+1}-\operatorname{sech}\alpha(x_r+x_{r+2})=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ 2x_{n-2}-\operatorname{sech}\alpha(x_{n-3}+x_{n-1})=0\\ 2x_{n-1}-\operatorname{sech}\alpha\,.\,x_{n-2}=0\end{array}\right.$$

Multiplying these respectively by $x_1, x_2, \ldots x_{n-1}$, and adding, we get

$$-2kV+2c=0\,;$$

$$\therefore\quad V=c/k \text{ is the maximum value required.}$$

From the first $(r-1)$ equations find $x_2, x_3, \ldots x_r$, in terms of x_1;

then $$x_s=Ae^{s\alpha}+Be^{-s\alpha}=x_1\sinh s\alpha/\sinh\alpha$$

for integral values of s from 1 to r.

The last $(n-r-1)$ equations give us

$$x_{n-\rho}=x_{n-1}\sinh\rho\alpha/(\sinh\alpha)$$

for integral values of ρ from 1 to $n-r$;

$$\therefore\quad x_r=x_{n-1}\sinh(n-r)\alpha/\sinh\alpha=x_1\sinh r\alpha/\sinh\alpha,$$

giving $$x_{n-\rho}=x_1\sinh\rho\alpha\,.\,\sinh r\alpha/\sinh(n-r)\alpha$$

for above values of ρ.

Substituting these values in the rth equation, we have for k,

$$2x_1\frac{\sinh.r\alpha}{\sinh.\alpha}\left(\frac{c}{V}-1\right)+\operatorname{sech}\alpha\left\{x_1\frac{\sinh(r-1)\alpha}{\sinh\alpha}+x_1\frac{\sinh(n-r-1)\alpha\,.\,\sinh r\alpha}{\sinh\alpha\,.\,\sinh(n-r)\alpha}\right\}=0,$$

easily reducing to

$$V=2c\sinh(n-r)\alpha\,.\,\sinh r\alpha\,.\,\cosh\alpha/\{\sinh n\alpha\sinh\alpha\}$$

as maximum value, the minimum being zero.

(2) Putting $V=\Sigma x_r^2$, the turning values of V are given by

$$-2Jx_r+2x_r-\operatorname{sech}\alpha(x_{r+1}+x_{r-1})=0,\ \text{ or }\ 2x_r-\operatorname{sech}\beta(x_{r+1}+x_{r-1})=0,$$

where, solving the system above, $x_r=x_1\sinh r\beta/\sinh\beta$, and from the last equation of all we have

$$2\frac{\sinh(n-1)\beta}{\sinh\beta}-\operatorname{sech}\beta\,.\,\frac{\sinh n\beta}{\sinh\beta}=0,\ \text{reducing to}\ \frac{\sinh n\beta}{\sinh\beta}=0,$$

thus the turning values are the solutions of $\sinh n\beta=0$, other than $\beta=0$, and the values of $\cosh\beta$ that we are concerned with are given by

$$\cosh\beta=\cos\frac{\pi}{n},\ \cos\frac{2\pi}{n},\ \cos\frac{3\pi}{n},\ \ldots\cos\frac{(n-1)\pi}{n},$$

where, however, an uncertainty attaches to $\cosh\beta=\cos\frac{\pi}{2}=0$ when n is even, but it is seen on reference to the system of equations for J that these are satisfied when n is even by $J=1$ corresponding to $\cosh\beta=0$; thus the turning values of V are

$$\frac{c\cosh\alpha}{\cosh\alpha\pm\cos\frac{\pi}{n}},\quad \frac{c\cosh\alpha}{\cosh\alpha\pm\cos\frac{2\pi}{n}}\cdots\frac{c\cosh\alpha}{\cosh\alpha\pm\cos\frac{r\pi}{n}}\cdots,$$

there being $(n-1)$ of these values including c when n is even. The absolutely greatest and least of these are

$$\frac{c\cosh\alpha}{\cosh\alpha\pm\cos\frac{\pi}{n}},$$

which should therefore replace $\frac{c\cosh\alpha}{\cosh\alpha\pm1}$ in the question.

Note 1. Some similar results are given in Wolstenholm's *Problems*, 218-221 inclusive. They are all easily established, but in 218, I think, $\dfrac{n}{2(n+1)}$ should be read for $\dfrac{n-1}{2n}$.

Note 2. Here, too, the analogy of the ellipsoid shews that there are only two true maxima and minima, the others being merely turning values as in the case of the b in the standard ellipsoid, $x^2/a^2+y^2/b^2+z^2/c^2=1$, where

$a^2>b^2>c^2$, a^2 is the maximum value of $r^2=x^2+y^2+z^2$,
c^2 is the minimum " " "
and b^2 is merely a turning value.

Part (3) follows in precisely the same way, the solution of the difference equations being quite mechanical after what has preceded.

276. *Trace the curves*

$$(a)\ y^3-y^2+4x^2-27x^4=0, \qquad (b)\ a(x^2+y^2)=xy(x+y);$$

and show that the circle $x^2+y^2+2a(1+\sqrt{2})(x+y-2a)=2a^2$ *has treble contact with the latter.* (C.)

Solution by J. F. Hudson.

(a) Curve $$y^3-y^2+4x^2-27x^4=0.$$

Tangents at origin $y=\pm 2x$. Curve cuts Ox at $(0, 0)$ and A, A' $\left(\pm\dfrac{2\sqrt{3}}{9}, 0\right)$, and cuts Oy at $(0, 0)$ and B $(0, 1)$. The points $\left(\pm\dfrac{2\sqrt{3}}{9}, 1\right)$ are on the curve, which is symmetrical about OY. No linear asymptote; form at infinity is given by $y^3-27x^4=0$.

If $x^2 \gtreqless \frac{4}{27}$, $y \gtreqless 1$. Hence PA, PQ, QA', easily seen to be tangents at A, B, A' respectively, each touch the curve on the side near the origin. Curve, being $(y-\frac{2}{3})^2(y+\frac{1}{3})=27(x^2-\frac{2}{27})^2$, is imaginary for $y>\frac{1}{3}$ and negative. Tangents at $(\pm\sqrt{\frac{2}{27}}, \frac{1}{3})$ are parallel to Ox; points $(\pm\sqrt{\frac{2}{27}}, \frac{2}{3})$ are multiple points, the tangents at which are $(y-\frac{2}{3})^2=8(x\mp\sqrt{\frac{2}{27}})^2$. The tangents at origin cut curve again in one point only, $y-2x=0$ at $(\frac{8}{27}, \frac{16}{27})$, and $y+2x=0$ at $(-\frac{8}{27}, \frac{16}{27})$.

Near the origin, $2x=\pm(y-\frac{1}{2}y^2)$, so that the curve is above the tangents at the origin. Transfer origin to P, tangent at P is $y=\dfrac{16}{\sqrt{27}}x$, and first approximation to form of curve at P is $y=\dfrac{16}{\sqrt{27}}x+\dfrac{28}{27}x^2$.

(b) Curve $$a(x^2+y^2)=xy(x+y).$$

Origin a conjugate point. $A(a, a)$ is on curve, and $x+y-2a=0$ is tangent at A. Curve is symmetrical about OA, for

$$2a[(x+y)^2+(x-y)^2]=[(x+y)^2-(x-y)^2](x+y).$$

Linear asymptotes, $x=a$, $y=a$, $x+y+2a=0$, curve cutting them at A and at infinity.

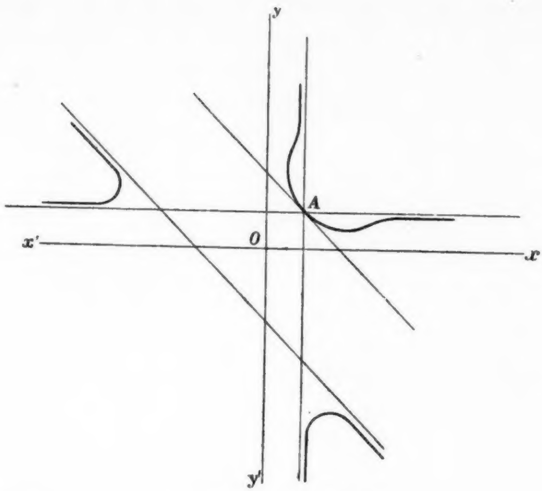
Transferring to A as origin, $a^2(x+y)+xy(x+y)+4axy=0$.

Form at new origin, $a^2(x+y)=4ax^2$, $y=-x+\dfrac{4x^2}{a}$;

$\therefore$ curve above the tangent.

The joins of the origin to the intersections of (b) and the circle are given by

$$(x^2+y^2)+\frac{2xy}{x^2+y^2}(1+\sqrt{2})(x+y)^2-2(1+\sqrt{2})^2(x+y)^2\frac{x^2y^2}{x^2+y^2}=0,$$



or for $x=\lambda y$, $$(\lambda-1)^2[(\lambda+1)^2+\lambda\sqrt{2}]^2=0:$$

$\therefore$ treble contact where $x=y$ and $(x+y)^2+xy\sqrt{2}=0$ cut the curve.

285. [I. i.] *Let l be the recurring period of r digits of the decimal equivalent to the vulgar fraction $\frac{1}{n}$. Let $n\kappa=p\,.\,10^q-1$, where κ is a number of 9 digits $(9<r)$. Show that* (1) *l terminates with the group κ;* (2) *the remaining digits may be found by continuously multiplying by p.*

[*Thus*, $29\times 31=900-1$. *The period for $\frac{1}{29}$ terminates with* 31, *and the remaining digits of the period can be found by multiplying continuously by* 9.]

E. M. Langley.

Solution by G. N. Bates.

(This question is not very clearly worded.)

Suppose $\dfrac{1}{nr-1}$ be represented by the radix fraction $\dfrac{a_1}{r}+\dfrac{a_2}{r^2}+\ldots+\dfrac{a_s}{r^s}$

then $$r^s-1=(nr-1)(a_1r^{s-1}+\ldots+a_{s-1}r+a_s).$$

Determining the values of a^s, etc., subject to the condition that they are all integral, and $0 \not> a_f \not> r-1$, we have

$$a_s=1,\quad a_{s-1}=n-m_1r,\quad a_{s-2}=na_{s-1}-m_2r+m_1,\ \text{etc.},$$

where m_1r, m_2r are so chosen that the above conditions are satisfied.

Hence $\dfrac{1}{nr-1}$ can be converted into a radix fraction in scale r by continuously multiplying by n.

In the present case $\dfrac{1}{p\,.\,10^q-1}$ will have its recurring period ending in 1 preceded by $(q-1)$ zeros;

$\therefore$ $\dfrac{1}{n}$ will have its recurring period ending with the group k, though we

must still make up a group of q figures by means of zeros before we can begin multiplying by p, *e.g.*

$$\frac{1}{43}=\frac{93}{4.10^3-1}, \text{ and } \frac{1}{4.10^3-1}=\cdot 0002 \ldots 6004001\ ;\ \frac{1}{43}=\cdot 02 \ldots 372093.$$

Solution by Proposer.

$$\frac{l}{10^r-1}=\frac{k}{p.10^q-1}=\frac{k(1+p.10^q+p^2.10^{2q}\ldots+p^{s-1}10^{(s-1)q})}{p^s.10^{qs}-1}.$$

Hence each $$=\frac{k(1+p.10^q+p^2.10^{2q}\ldots+p^{s-1}10^{(s-1)q})-l}{p^s.10^{qs}-10^r}.$$

Hence if qs be taken greater than r, the numerator must contain 10^r as a factor, and l must be the number formed by the last r digits of

$$k(1+p10^q+p^2.10^{2q}+\ldots).$$

The arithmetical application is easy. Taking the example given, any number of the digits beginning from the end of the period for $\frac{1}{2q}$ can be found by adding up the numbers 31, 27900, 25110000 .... But the multiplication and addition may be performed simultaneously. Write down 31, then say 9 times 1, 9, which write to left of 3; 9 times 3, 27, write down 7 and carry 2; 9 times 9, 81, and 2 to carry, 83, put down 3, and so on (... 37931).

In the above, n was not supposed to contain 2 or 5 as a factor; and in the question 9 has been printed for q. If the group k contained more than q digits the method would still apply though the enunciation would require a slight modification.

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* Will be reviewed shortly.

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